

HYPERCONTRACTIVITY ON THE q -ARAKI-WOODS ALGEBRAS

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ABSTRACT. Extending a work of Carlen and Lieb, Biane has obtained the optimal hypercontractivity of the q -Ornstein-Uhlenbeck semigroup on the q -deformation of the free group algebra. In this note, we look for an extension of this result to the type III situation, that is for the q -Araki-Woods algebras. We show that hypercontractivity from L^p to L^2 can occur if and only if the generator of the deformation is bounded.

1. INTRODUCTION

In [14] Nelson proved the following famous hypercontractivity result for the classical Ornstein-Uhlenbeck semigroup P_t^1 acting on $L^p(\mathbb{R}^n, d\gamma)$, where $d\gamma$ is the n -dimensional gaussian measure on \mathbb{R}^n .

Theorem 1.1. (Nelson, 1973)

For $1 < p < r < \infty$ we have

$$\|P_t^1\|_{L^p \rightarrow L^r} \leq 1 \text{ if and only if } e^{-2t} \leq \frac{p-1}{r-1}.$$

Since then, there have been several analogous results in the context of non-commutative probability. A fermionic counterpart of the Nelson's result has been clarified by Carlen/Lieb in [8], and including the fermionic case Biane proved in [3] the following hypercontractivity result for the q -Ornstein-Uhlenbeck semigroup P_t^q acting on $L^p(\Gamma_q, \tau_q)$, where Γ_q is the von Neumann algebra generated by q -gaussians by Bożejko and Speicher ([6, 7]) and τ_q is the vacuum state.

Theorem 1.2. (Biane, 1997)

Let $-1 \leq q < 1$. For $1 < p < r < \infty$ we have

$$\|P_t^q\|_{L^p \rightarrow L^r} \leq 1 \text{ if and only if } e^{-2t} \leq \frac{p-1}{r-1}.$$

The above result has been further extended to the case of gaussians satisfying more general commutation relations by Krolak ([13]), and the holomorphic version has been proved by Kemp ([11]).

Biane's generalized result concerns only about von Neumann algebras with a normal tracial state. Thus, it is natural to be interested in their non-tracial relatives, namely, q -Araki-Woods algebras $\Gamma_q(H_{\mathbb{R}}, (U_t))$ ([9] or see section 2), which is a generalization of Araki-Woods factors ([1]) depending on the deformation group of orthogonal transformations (U_t) on some real Hilbert space $H_{\mathbb{R}}$. As usual, we denote by A the (unbounded) generator of (U_t) on the complexification of $H_{\mathbb{R}}$. Fortunately, we were able to prove the following hypercontractivity result for the q -Ornstein-Uhlenbeck semigroup $P_t^{q, (U_t)}$ (simply P_t^q again) acting on $L^p(\Gamma_q(H_{\mathbb{R}}, (U_t)), \tau_q)$, where τ_q is the vacuum state.

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Theorem 1.3. *Let $-1 \leq q < 1$. For $1 < p < 2$ we have*

$$\|P_t^q\|_{L^p \rightarrow L^2} \leq 1 \text{ if } e^{-2t} \leq C \|A\|^{1-\frac{2}{p}} (p-1),$$

where C is a universal constant.

If A is unbounded, then P_t^q is not bounded from L^p to L^2 .

Following the idea of Carlen/Lieb and Biane we will use the baby Fock model and Speicher's central limit procedure as key ingredients for the proof. Although we were unable to determine "the optimal time" for the contractivity, the constant in the above shows the same optimal order $p-1$ as in the tracial case.

This paper is organized as follows. In section 2 we will review about q -generalized gaussians, q -Araki-Woods algebras, and q -Ornstein-Uhlenbeck semigroup. We will also briefly review of an extension procedure of a map on non-tracial von Neumann algebras to the associated non-commutative L^p spaces. In section 3 we will introduce the *twisted baby Fock* model by Nou ([16]), and we will prove the first main result, hypercontractivity of ε -Ornstein-Uhlenbeck semigroup, which is an analog of q -Ornstein-Uhlenbeck semigroup in the baby Fock model. In section 4 we will use Speicher's central limit procedure and hypercontractivity of ε -Ornstein-Uhlenbeck semigroup to prove hypercontractivity of the q -Ornstein-Uhlenbeck semigroup. Finally in section 5 we will use 1-dimensional estimate to get the converse direction of the main result from the previous section.

2. q -ARAKI-WOODS ALGEBRAS AND q -ORNSTEIN-UHLENBECK SEMIGROUP

We begin with the most general definition of q -Araki-Woods algebra (see [1, 9, 19]). The construction starts with a (separable) real Hilbert space $H_{\mathbb{R}}$ and $(U_t)_{t \in \mathbb{R}}$ a group of orthogonal transformations on $H_{\mathbb{R}}$. Let $H_{\mathbb{C}} = H_{\mathbb{R}} + iH_{\mathbb{R}}$ be the complexification of $H_{\mathbb{R}}$. The group (U_t) extends to a group of unitaries on $H_{\mathbb{C}}$. By Banach-Stone's theorem, there is a self-adjoint (unbounded) operator A so that $U_t = A^{it}$. One then defines a new scalar product on $H_{\mathbb{C}}$ by

$$\langle x, y \rangle_U = \left\langle \frac{2A}{1+A} x, y \right\rangle_{H_{\mathbb{C}}}.$$

Its completion is then denoted by \mathcal{H} .

We consider the operator of symmetrization P_n on $\mathcal{H}^{\otimes n}$ defined by

$$P_0 \Omega = \Omega, \\ P_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)},$$

where S_n denotes the symmetric group of permutations of n elements and

$$i(\pi) = \#\{(i, j) | 1 \leq i, j \leq n, \pi(i) > \pi(j)\}$$

is the number of inversions of $\pi \in S_n$.

Now we define the q -inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{H}^{\otimes n}$ by

$$\langle \xi, \eta \rangle_q = \delta_{n,m} \langle \xi, P_n \eta \rangle \text{ for } \xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}^{\otimes n}$. We denote by $\mathcal{H}^{\otimes_q n}$ the resulting Hilbert space. Since P_n 's are strictly positive for $-1 < q < 1$ ([6]), $\langle \cdot, \cdot \rangle_q$ is actually an inner product. Then one can associate a q -Fock space $\mathcal{F}_q(\mathcal{H})$.

$$\mathcal{F}_q(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes_q n},$$

where Ω is a unit vector called vacuum.

When $q = 0$ we recover the classical full Fock space over \mathcal{H} . In the extreme cases $q = \pm 1$, $\mathcal{F}_1(\mathcal{H})$ and $\mathcal{F}_{-1}(\mathcal{H})$ refer to Bosonic and Fermionic Fock spaces, respectively. In the whole paper we are interested in $-1 \leq q < 1$.

For $h \in H_{\mathbb{R}}$, we can define a generalized q -semi-circular random variable on $\mathcal{F}_q(\mathcal{H})$ by

$$s(h) = \ell(h) + \ell^*(h),$$

where $\ell_q(h)$ is the left creation operator by $h \in \mathcal{H}$ and $\ell_q^*(h)$ is the adjoint of $\ell_q(h)$.

By definition $\Gamma_q(H_{\mathbb{R}}, (U_t)) = \{s(h) : h \in H_{\mathbb{R}}\}''$. The vacuum state τ_q defined by $\tau_q(\cdot) = \langle \cdot, \Omega, \Omega \rangle_q$ is a normal faithful state on $\Gamma_q(H_{\mathbb{R}}, (U_t))$, and $(\Gamma_q(H_{\mathbb{R}}, (U_t)), \tau_q, \Omega)$ is in GNS position. $\Gamma_q(H_{\mathbb{R}}, (U_t))$ is known to be a type II_1 algebra if and only if (U_t) is trivial, in general, it is a type III von Neumann algebra, whose modular theory relative to τ_q is well understood ([19, 9]).

Now we consider the q -Ornstein-Uhlenbeck semigroup $P_t^{q, (U_t)}$ (simply P_t^q). Since Ω is a separating vector for $\Gamma_q(H_{\mathbb{R}}, (U_t))$ we can use the second quantization ([9, Proposition 1.1], [5, Theorem 2.11]) to get the semigroup $P_t^q : \Gamma_q \rightarrow \Gamma_q$ given by

$$P_t^q(X)\Omega = \mathcal{F}(e^{-t}id_{\mathcal{H}})X\Omega,$$

where $\mathcal{F}(e^{-t}id_{\mathcal{H}}) : \mathcal{F}_q(\mathcal{H}) \rightarrow \mathcal{F}_q(\mathcal{H})$ defined by

$$\mathcal{F}(e^{-t}id_{\mathcal{H}})|_{\mathcal{H}^{\otimes n}} = e^{-nt}id_{\mathcal{H}^{\otimes n}}, \quad n \geq 0.$$

Note that second quantizations commute with the modular group of τ_q . For the later use we record the properties of the semigroup as follows.

Proposition 2.1. *P_t^q , $t \geq 0$ is a completely positive, normal, τ_q -preserving contraction that commutes with the modular group of τ_q .*

Recall that A is said to be almost periodic when it has an orthonormal basis of eigenvectors. In this situation, there is a more tractable model for $\Gamma_q(H_{\mathbb{R}}, (U_t))$ that we will use (see Section 2.2 in [16]).

Assume that we are given a sequence $\mu = (\mu_i)_{i \geq 1} \subseteq [1, \infty)$. The construction starts with a separable complex Hilbert space \mathcal{H} equipped with an orthonormal basis $(e_{\pm k})_{k \geq 1}$. We denote as above by $\mathcal{F}_q(\mathcal{H})$ the associated q -Fock space. We define q -(generalized) gaussian variables (or q -generalized circular variables) by

$$g_{q,i} = \mu_i^{-1} \ell_q(e_i) + \mu_i \ell_q^*(e_{-i}).$$

Let Γ_q^μ ($-1 \leq q < 1$), the von Neumann algebra generated by $\{g_{q,k}\}_{k \geq 1}$ and τ_q still denotes the vacuum state.

If A is almost periodic and $(\mu_i^4)_{i \geq 1} \subseteq [1, \infty)$ is the sequence of eigenvalues of A that are bigger than 1, then there is a spatial isomorphism

$$(\Gamma_q(H_{\mathbb{R}}, (U_t)), \tau_q, \Omega) \cong (\Gamma_q^\mu, \tau_q, \Omega).$$

Of course, one has $\|A\| = \sup \mu_i^4$.

In this model, the q -Ornstein-Uhlenbeck semigroup $P_t^{q, \mu}$ (simply P_t^q) is also obtained by the second quantization procedure with the same formula.

All the properties we will be looking at are stable under taking ultraproducts, so that the discretization procedure in Section 6.1 of [16] allows us to work only in the almost periodic situation.

In this paper we will often need an extension of a map defined on the algebra level to the L^p setting. Let \mathcal{M}, \mathcal{N} be von Neumann algebras with distinguished normal faithful states φ and ψ , respectively. Then, one can define the Haagerup's L^p space $L^p(\mathcal{M}, \varphi)$, $1 \leq p < \infty$. For the details about $L^p(\mathcal{M}, \varphi)$ (simply $L^p(\mathcal{M})$) we refer [10, 17]. Let $T : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive contraction. We say T is *state-preserving* if $\psi \circ T = \varphi$ and it intertwines the modular groups: $\sigma_t^\psi \circ T = T \circ \sigma_t^\varphi$ for all $t \in \mathbb{R}$. Let $\text{tr}_{\mathcal{M}}$ be the trace on $L^1(\mathcal{M})$ and D_φ be the density operator associated to φ . Similarly, we consider $\text{tr}_{\mathcal{N}}$ and D_ψ . Then $\mathcal{M}D_\varphi^{\frac{1}{p}}$ is norm-dense in $L^p(\mathcal{M})$, and for $x \in \mathcal{M}$ the elements $x D_\varphi^{\frac{1}{p}} \in L^p(\mathcal{M})$ ($1 \leq p < \infty$) are identified in the sense of complex interpolation. Now we consider an extension of T to L^p

setting given by $T^p : \mathcal{M}D_\varphi^{\frac{1}{p}} \rightarrow \mathcal{N}D_\psi^{\frac{1}{p}}$, $x D_\varphi^{\frac{1}{p}} \mapsto (Tx)D_\psi^{\frac{1}{p}}$. It is well-known that T^p can be extended to a contraction on $L^p(\mathcal{M})$ ([10, Lemma 2.2]). For a later use we record the extension procedure as follows.

Proposition 2.2. *Let \mathcal{M}, \mathcal{N} be von Neumann algebras with distinguished normal faithful states φ and ψ , respectively, and $T : \mathcal{M} \rightarrow \mathcal{N}$ be a state-preserving completely positive contraction. Then*

$$(2.1) \quad T^p : \mathcal{M}D_\varphi^{\frac{1}{p}} \rightarrow \mathcal{N}D_\psi^{\frac{1}{p}}, \quad x D_\varphi^{\frac{1}{p}} \mapsto (Tx)D_\psi^{\frac{1}{p}}.$$

extends to a contraction on $L^p(\mathcal{M})$.

In particular, if T is an onto isometry, then so is T^p between $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$.

Inspired by the above we can consider the following further extension of T . Let $1 \leq p, r < \infty$.

$$(2.2) \quad T^{p,r} : \mathcal{M}D_\varphi^{\frac{1}{p}} \rightarrow \mathcal{N}D_\psi^{\frac{1}{r}} \subseteq L_r(\mathcal{N}), \quad x D_\varphi^{\frac{1}{p}} \mapsto (Tx)D_\psi^{\frac{1}{r}}.$$

In general, there is no guarantee that $T^{p,r}$ can be extended to a bounded map from $L^p(\mathcal{M})$ into $L_r(\mathcal{N})$ when $r > p$.

If we apply the above extension (2.1) to the q -Ornstein-Uhlenbeck semigroup P_t^q , we obtain a contractive semigroup $P_t^{q,p}$ on $L^p(\Gamma_q)$.

$$P_t^{q,p} : L^p(\Gamma_q) \rightarrow L^p(\Gamma_q), \quad x D_q^{\frac{1}{p}} \mapsto P_t^q(x)D_q^{\frac{1}{p}},$$

where D_q is the density operator associated to the vacuum state τ_q .

Now the question is for which $1 < p < r < \infty$ and for which $t > 0$, can we extend P_t^q to a contraction from $L^p(\Gamma_q)$ into $L^r(\Gamma_q)$? More precisely, when does the map

$$(2.3) \quad P_t^{q,p,r} : \Gamma_q D_q^{\frac{1}{p}} \rightarrow \Gamma_q D_q^{\frac{1}{r}}, \quad x D_q^{\frac{1}{p}} \mapsto (P_t^q x)D_q^{\frac{1}{r}}$$

can be extended to a contraction from $L^p(\Gamma_q)$ into $L^r(\Gamma_q)$? Here comes a partial answer to this question, which is one of our main results. Note that we will simply denote $P_t^{q,p,r}$ again by P_t^q .

Theorem 2.3. *Let $1 < p < 2$ and P_t^q is the map in (2.3). Then, we have*

$$\|P_t^q\|_{L^p \rightarrow L^2} \leq 1 \quad \text{if} \quad e^{-2t} \leq C \alpha_\mu^{4-\frac{8}{p}} (p-1)$$

for some universal constant $C > 0$, where $\alpha_\mu = \sup_{n \geq 1} \mu_n$.

We close this section with some precise results on the modular theory for Γ_q and τ_q that we need. The modular group $\sigma_t^{\tau_q}$ with respect to τ_q satisfies the following.

$$\sigma_t^{\tau_q}(g_{q,k}) = \mu_k^{4it} g_{q,k}, \quad k \geq 1.$$

Thus, $g_{q,k}$ is an analytic element satisfying

$$(2.4) \quad D_q^{\frac{1}{2p}} g_{q,k} = \mu_k^{\frac{2}{p}} g_{q,k} D_q^{\frac{1}{2p}}, \quad k \geq 1.$$

3. TWISTED BABY FOCK AND HYPERCONTRACTIVITY OF ε -ORNSTEIN-UHLENBECK SEMIGROUP

3.1. The baby Fock model. We will briefly describe the twisted baby Fock introduced by A. Nou ([16]). Let $I = \{\pm 1, \pm 2, \dots, \pm n\}$ be a fixed index set and $\varepsilon : I \times I \rightarrow \{\pm 1\}$ be a “choice of sign” function satisfying

$$\varepsilon(i, j) = \varepsilon(j, i), \quad \varepsilon(i, i) = -1, \quad \varepsilon(i, j) = \varepsilon(|i|, |j|), \quad \forall i, j \in I.$$

Now, we consider the unital algebra $\mathcal{A}(I, \varepsilon)$ with generators $(x_i)_{i \in I}$ satisfying

$$x_i x_j - \varepsilon(i, j) x_j x_i = 2\delta_{i,j}, \quad i, j \in I.$$

In particular, we have $x_i^2 = 1$, $i \in I$, where 1 refers to the unit of the algebra. $\mathcal{A}(I, \varepsilon)$ can be endowed with the involution given by $x_i^* = x_i$. We will use the following notations for the elements in $\mathcal{A}(I, \varepsilon)$.

$$x_\emptyset := 1 \text{ and } x_A := x_{i_1} \cdots x_{i_k}, \quad A = \{i_1 < \cdots < i_k\} \subseteq I.$$

Then, $\{x_A : A \subseteq I\}$ is a basis for $\mathcal{A}(I, \varepsilon)$. Let $\phi^\varepsilon : \mathcal{A}(I, \varepsilon) \rightarrow \mathbb{C}$ be the tracial state given by

$$\phi^\varepsilon(x_A) = \delta_{A, \emptyset}.$$

ϕ^ε give rise to a natural inner product on $\mathcal{A}(I, \varepsilon)$ as follows.

$$\langle x, y \rangle := \phi^\varepsilon(y^* x), \quad x, y \in \mathcal{A}(I, \varepsilon).$$

Let

$$H = L^2(\mathcal{A}(I, \varepsilon), \phi^\varepsilon)$$

be the corresponding L^2 -space, then clearly $\{x_A : A \subseteq I\}$ is an orthonormal basis for H .

Now we consider left creations β_i^* and left annihilations β_i in $B(H)$, $i \in I$ in this context.

$$\beta_i^*(x_A) = \begin{cases} x_i x_A & \text{if } i \notin A \\ 0 & \text{if } i \in A \end{cases}, \quad \beta_i(x_A) = \begin{cases} x_i x_A & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}, \quad i \in I, \quad A \subseteq I.$$

Then using the same parameter $\mu = (\mu_i)_{i \geq 1}$ we define our generalized gaussians

$$\gamma_i := \mu_i^{-1} \beta_i^* + \mu_i \beta_{-i}, \quad 1 \leq i \leq n.$$

The following relations are known to be satisfied by γ_i 's ([16, Lemma 5.2]).

$$(3.1) \quad \begin{cases} \gamma_i \gamma_j - \varepsilon(i, j) \gamma_j \gamma_i = 0 & i \neq j \in I \\ \gamma_i^* \gamma_j - \varepsilon(i, j) \gamma_j \gamma_i^* = 0 & i \neq j \in I \\ \gamma_i^2 = (\gamma_i^*)^2 = 0 & i \in I \\ \gamma_i^* \gamma_i + \gamma_i \gamma_i^* = (\mu_i^2 + \mu_i^{-2}) id & i \in I \end{cases}$$

Let $\Gamma_{\langle 1, \dots, n \rangle}$ be the von Neumann algebra generated by $\{\gamma_i\}_{i=1}^n$ in $B(H)$ while $\Gamma_{\langle n \rangle}$ refers to the von Neumann algebra generated by γ_n . It is also known that ([16, Lemma 5.2]) 1 is a cyclic and separating vector for $\Gamma_{\langle 1, \dots, n \rangle}$, and the above is the faithful GNS representation of $(\Gamma_{\langle 1, \dots, n \rangle}, \tau_n^\varepsilon)$, where τ_n^ε is the vacuum state on $\Gamma_{\langle 1, \dots, n \rangle}$ given by $\tau_n^\varepsilon(\cdot) = \langle \cdot 1, 1 \rangle$. With this definition, $\Gamma_{\langle 1, \dots, k \rangle}$ is not a subalgebra of $\Gamma_{\langle 1, \dots, n \rangle}$, but by the above facts we can indeed identify it with the subalgebra generated by $\gamma_1, \dots, \gamma_k$ in $\Gamma_{\langle 1, \dots, n \rangle}$ and τ_k^ε is then the restriction of τ_n^ε . We may sometimes simply write τ^ε without any reference to n . Finally, we remark that with the classical identification $x \mapsto x1$ between $\Gamma_{\langle 1, \dots, n \rangle}$ and H , τ^ε corresponds to ϕ^ε .

We collect some results about γ_i 's which we need in the sequel.

Proposition 3.1. *We have:*

- (1) For all $n \geq 1$, and $i \leq n$, $\sigma_t^{\tau^\varepsilon}(\gamma_i) = \mu_i^{4it} \gamma_i$ and $\tau^\varepsilon(\gamma_i^* \gamma_i) = \mu_i^{-2}$.
- (2) Let $D_{\langle 1, \dots, n \rangle}$ and $D_{\langle 1, \dots, n-1 \rangle}$ be the densities of τ^ε restricted to $\Gamma_{\langle 1, \dots, n \rangle}$ and $\Gamma_{\langle 1, \dots, n-1 \rangle}$, respectively. Then we have a natural isometric embedding

$$L^p(\Gamma_{\langle 1, \dots, n-1 \rangle}) \hookrightarrow L^p(\Gamma_{\langle 1, \dots, n \rangle}), \quad x D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \mapsto x D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}.$$

A similar statement holds for $L^p(\Gamma_{\langle n \rangle})$.

- (3) For any $a \in \Gamma_{\langle n \rangle}$ and $b, c \in \Gamma_{\langle 1, \dots, n-1 \rangle}$ we have

$$\tau^\varepsilon(cab) = \tau^\varepsilon(a) \tau^\varepsilon(cb).$$

- (4) $\gamma_n^* \gamma_n$ commutes with $\Gamma_{\langle 1, \dots, n-1 \rangle}$ and $D_{\langle 1, \dots, n \rangle}$.

Proof. The computations for the first point can be found in [16, Proposition 5.3]. Thus, the natural inclusion of $\Gamma_{\langle 1, \dots, n-1 \rangle}$ to $\Gamma_{\langle 1, \dots, n \rangle}$ is state preserving and one can use Proposition 2.2. For the third point, it suffices to do it with $c = 1$ using the modular theory, but then it is [16, Lemma 5.4]. Finally for $i < n$

$$\gamma_n^* \gamma_n \gamma_i = \gamma_n^* (\varepsilon(n, i) \gamma_i \gamma_n) = \varepsilon(n, i)^2 \gamma_i \gamma_n^* \gamma_n = \gamma_i \gamma_n^* \gamma_n.$$

And $\sigma_t^\varepsilon(\gamma_n^* \gamma_n) = \sigma_t^{\tau^\varepsilon}(\gamma_n)^* \sigma_t^{\tau^\varepsilon}(\gamma_n) = \gamma_n^* \gamma_n$, so $\gamma_n^* \gamma_n$ is in the centralizer of τ^ε which exactly means that it commutes with $D_{\langle 1, \dots, n \rangle}$. \square

Remark 3.2. Note that we are dealing with finite dimensional algebras, so we know that the density $D_{\langle 1, \dots, n \rangle}$ belongs to $\Gamma_{\langle 1, \dots, n \rangle}$. Moreover all associated L^p -spaces can be identified with a p -Schatten classes provided that we fix a faithful trace on $\Gamma_{\langle 1, \dots, n \rangle}$ (instead of looking at the trace on the dual space $\Gamma_{\langle 1, \dots, n \rangle}^*$).

The main estimates relies on the position of $\Gamma_{\langle 1, \dots, n-1 \rangle}$ inside $\Gamma_{\langle 1, \dots, n \rangle}$, which will be clarified in the following Proposition. Now we set

$$(3.2) \quad y_i = \gamma_i^* \gamma_i - \mu_i^{-2} id, \quad 1 \leq i \leq n$$

In this paper we will consider the basis $\{1, \gamma_i, \gamma_i^*, y_i\}$ of $\Gamma_{\langle i \rangle}$ which is more suitable than $\{\gamma_i \gamma_i^*, \gamma_i, \gamma_i^*, \gamma_i^* \gamma_i\}$, $1 \leq i \leq n$, as their corresponding L^2 vectors are orthogonal and have length 0, 1, 1 and 2, respectively. Then, for a fixed $1 \leq i \leq n$, any element $X \in \Gamma_{\langle 1, \dots, n \rangle}$ can be *uniquely* expressed in the form

$$(3.3) \quad X = a + \gamma_i b + \gamma_i^* c + y_i d$$

for some $a, b, c, d \in \Gamma_{\langle 1, \dots, n \rangle \setminus i}$, the von Neumann algebra generated by $\{\gamma_k : 1 \leq k \neq i \leq n\}$.

Proposition 3.3. *There are a $*$ -isomorphism $\Phi : \Gamma_{\langle 1, \dots, n \rangle} \rightarrow \mathbb{M}_2 \otimes \Gamma_{\langle 1, \dots, n-1 \rangle}$ and a unitary $u_n \in \Gamma_{\langle 1, \dots, n-1 \rangle}$ satisfying the following.*

- (1) For $a \in \Gamma_{\langle 1, \dots, n-1 \rangle}$ we have $\Phi(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.
- (2) $\Phi(\gamma_n) = (\mu_n^2 + \mu_n^{-2})^{\frac{1}{2}} \begin{bmatrix} 0 & 0 \\ u_n & 0 \end{bmatrix}$.
- (3) $\tau_n^\varepsilon = (\psi \otimes \tau_{n-1}^\varepsilon) \Phi$, where $\psi \left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \right) = \lambda x_{11} + (1 - \lambda) x_{22}$ with $\lambda = \frac{1}{1 + \mu_n^4}$.

Proof. The relations 3.1 give us a $*$ -isomorphism $\sigma : \Gamma_{\langle n \rangle} \rightarrow \mathbb{M}_2$ with $\sigma(\gamma_n) = (\mu_n^2 + \mu_n^{-2})^{1/2} e_{21}$. Let C_n be the relative commutant of $\Gamma_{\langle n \rangle}$ in $\Gamma_{\langle 1, \dots, n \rangle}$. Then, the multiplication map $\psi : \Gamma_{\langle n \rangle} \otimes C_n \rightarrow \Gamma_{\langle 1, \dots, n \rangle}$ is a $*$ -homomorphism. Actually, ψ is a $*$ -isomorphism. Indeed, we can easily check that ψ is an onto map. For example, we have

$$\psi((\varepsilon(1, n) \gamma_n^* \gamma_n + \gamma_n \gamma_n^*) \otimes \gamma_1 (\varepsilon(1, n) \gamma_n^* \gamma_n + \gamma_n \gamma_n^*)) = (\mu_n^2 + \mu_n^{-2})^2 \gamma_1.$$

The fact that $\gamma_1 (\varepsilon(1, n) \gamma_n^* \gamma_n + \gamma_n \gamma_n^*) \in C_n$ is a straightforward calculation. Moreover, ψ must be a 1-1 map, since $\dim C_n = 2^{2n-2}$, which can be checked by a simple induction and the following observation. Let $X = a + \gamma_1 b + \gamma_1^* c + y_1 d$, $a, b, c, d \in \Gamma_{\langle 2, \dots, n \rangle}$. Then by the uniqueness of the expression (3.3) we have

$$X \in C_n \Leftrightarrow a, d \in \Gamma_{\langle n \rangle}' \cap \Gamma_{\langle 2, \dots, n \rangle} \text{ and } \begin{cases} \varepsilon(1, n) \gamma_n b = b \gamma_n, & \varepsilon(1, n) \gamma_n^* b = b \gamma_n^* \\ \varepsilon(1, n) \gamma_n c = c \gamma_n, & \varepsilon(1, n) \gamma_n^* c = c \gamma_n^* \end{cases}.$$

Now we consider another $*$ -isomorphism $\pi = (\sigma \otimes Id) \circ \psi^{-1} : \Gamma_{\langle 1, \dots, n \rangle} \rightarrow \mathbb{M}_2 \otimes C_n$. Since $\gamma_n^* \gamma_n$ commutes with $\Gamma_{\langle 1, \dots, n-1 \rangle}$ there are $*$ -homomorphisms $\pi_i : \Gamma_{\langle 1, \dots, n-1 \rangle} \rightarrow e_{ii} \otimes C_n$, $a \mapsto (e_{ii} \otimes 1)\pi(a)(e_{ii} \otimes 1)$, $i = 1, 2$. Actually, π_i 's are $*$ -isomorphisms since $\dim \Gamma_{\langle 1, \dots, n-1 \rangle} = 2^{2n-2} = \dim C_n$ ([16, Lemma 5.2]) and they are injective. Indeed, we have

$$\begin{aligned} \pi^{-1}((e_{ii} \otimes id)\pi(a)(e_{ii} \otimes id)) &= (\mu_n^2 + \mu_n^{-2})^{-2} \gamma_n^* \gamma_n a \gamma_n^* \gamma_n \\ &= (\mu_n^2 + \mu_n^{-2})^{-1} \gamma_n^* \gamma_n a. \end{aligned}$$

Then, the injectivity comes from the uniqueness of the expression (3.3).

Now by identifying $e_{ii} \otimes C_n$ and C_n we get two $*$ -isomorphisms

$$\rho_1, \rho_2 : \Gamma_{\langle 1, \dots, n-1 \rangle} \rightarrow C_n \text{ such that } \pi(a) = \begin{bmatrix} \rho_1(a) & 0 \\ 0 & \rho_2(a) \end{bmatrix}.$$

The above $*$ -isomorphisms enable us to conclude that $\Gamma_{\langle 1, \dots, k \rangle} \cong \mathbb{M}_{2^k}$, $k \geq 1$ by a simple induction. Thus, any automorphism on $\Gamma_{\langle 1, \dots, n-1 \rangle}$ is inner, so that there is a unitary $u_n \in \Gamma_{\langle 1, \dots, n-1 \rangle}$ such that $\rho_1^{-1} \circ \rho_2(a) = u_n^* a u_n$, $a \in \Gamma_{\langle 1, \dots, n-1 \rangle}$. Finally we define $\Phi : \Gamma_{\langle 1, \dots, n \rangle} \rightarrow \mathbb{M}_2 \otimes \Gamma_{\langle 1, \dots, n-1 \rangle}$ by

$$\Phi(x) = \begin{bmatrix} 1 & 0 \\ 0 & u_n^* \end{bmatrix} [(I \otimes \rho_1^{-1}) \circ \pi](x) \begin{bmatrix} 1 & 0 \\ 0 & u_n \end{bmatrix}.$$

Then, the first two assertions easily follow from the construction of Φ and u_n . The formula for τ_n^ε is a consequence of Proposition 3.1 (3) and the definition of Φ .

For example, we have $\tau_n^\varepsilon(\gamma_n b) = 0 = \psi \otimes \tau_{n-1}^\varepsilon \left(\begin{bmatrix} 0 & 0 \\ (\mu_n^2 + \mu_n^{-2})^{\frac{1}{2}} u_n b & 0 \end{bmatrix} \right)$ for $b \in \Gamma_{\langle 1, \dots, n-1 \rangle}$, and $\tau_n^\varepsilon(y_n d) = 0 = \psi \otimes \tau_{n-1}^\varepsilon \left(\begin{bmatrix} \mu_n^2 d & 0 \\ 0 & -\mu_n^{-2} d \end{bmatrix} \right)$ for $d \in \Gamma_{\langle 1, \dots, n-1 \rangle}$. \square

Now we consider the number operator N_ε on H given by $N_\varepsilon = \sum_{i \in I} \beta_i^* \beta_i$. Then, for any $A \subseteq I$ we have $N_\varepsilon x_A = |A| x_A$. Since $1 \in H$ is separating and cyclic for $\Gamma_{\langle 1, \dots, n \rangle}$ we define the ε -Ornstein-Uhlenbeck semigroup $P_t^\varepsilon : \Gamma_{\langle 1, \dots, n \rangle} \rightarrow \Gamma_{\langle 1, \dots, n \rangle}$ by

$$(3.4) \quad P_t^\varepsilon(X)1 = e^{-tN_\varepsilon}(X1), \quad X \in \Gamma_{\langle 1, \dots, n \rangle}.$$

To make this definition more explicit, any element in $\Gamma_{\langle 1, \dots, n \rangle}$ can be written as a linear combination of products $w_1 \dots w_k$ where $w_i \in \{id, \gamma_i, \gamma_i^*, y_i\}$, $1 \leq i \leq k$. The number operator counts 0 for id , 1 for γ_i, γ_i^* and 2 for y_i ; for instance $P_t^\varepsilon(y_4 \gamma_2^* \gamma_1) = e^{-4t} y_4 \gamma_2^* \gamma_1$. This can be checked by a straightforward induction.

In comparison to the q -Fock space setting, it not easy to see that this defines a completely positive semigroup. There is no general second quantization in the baby Fock model. Nevertheless, such a procedure exists for some diagonal contractions. To do so, define similarly the i -number operator N_i on H and T_i^t on $\Gamma_{\langle 1, \dots, n \rangle}$ by

$$N_i = \beta_i^* \beta_i + \beta_{-i}^* \beta_{-i}, \quad 1 \leq i \leq n \text{ and } T_i^t(X)1 = e^{-tN_i}(X1).$$

It counts only the letter i as explained above.

Proposition 3.4. *For any $t \geq 0$, the operators T_i^t ($1 \leq i \leq n$) are completely positive and state preserving on $\Gamma_{\langle 1, \dots, n \rangle}$, and so is P_t^ε .*

Proof. The second assertion follows from the first one as $P_t^\varepsilon = T_1^t \dots T_n^t$.

For simplicity we only check the case $i = n$. Let $a, b, c, d \in \Gamma_{\langle 1, \dots, n-1 \rangle}$, we have $T_n^t(a) = a$, $T_n^t(\gamma_n b) = e^{-t} \gamma_n b$, $T_n^t(\gamma_n^* c) = e^{-t} \gamma_n^* c$ and $T_n^t(y_n d) = e^{-2t} y_n d$.

We use Proposition 3.3 to transfer T_n^t to $\tilde{T}_n^t = \Phi \circ T_n^t \circ \Phi^{-1}$ on $\mathbb{M}_2 \otimes \Gamma_{\langle 1, \dots, n \rangle}$.

From the formula for $\Phi(\gamma_n)$ it follows that $\tilde{T}_n^t \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = e^{-t} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and since

$$\Phi(a + y_n d) = \begin{bmatrix} a + \mu_n^2 d & 0 \\ 0 & a - \mu_n^{-2} d \end{bmatrix} \text{ for } a, d \in \Gamma_{\langle 1, \dots, n-1 \rangle} \text{ we have}$$

$$\tilde{T}_n^t \begin{bmatrix} a + \mu_n^2 d & 0 \\ 0 & a - \mu_n^{-2} d \end{bmatrix} = e^{-t} \begin{bmatrix} a + \mu_n^2 e^{-2t} d & 0 \\ 0 & a - \mu_n^{-2} e^{-2t} d \end{bmatrix}.$$

Thus, we obtain that $\tilde{T}_n^t = T \otimes Id$, where $T(e_{12}) = e^{-t} e_{12}$, $T(e_{21}) = e^{-t} e_{21}$, $T(1) = 1$ and $T(\mu_n^2 e_{11} - \mu_n^{-2} e_{22}) = e^{-2t} (\mu_n^2 e_{11} - \mu_n^{-2} e_{22})$. We get with $\lambda = \frac{1}{1+\mu_n^4}$

$$\begin{aligned} T(e_{11}) &= \lambda(1 + e^{-2t} \mu_n^4) e_{11} + \lambda(1 - e^{-2t}) e_{22} \\ T(e_{22}) &= (1 - \lambda)(1 - e^{-2t}) e_{11} + (1 - \lambda)(1 - e^{-2t} \mu_n^{-4}) e_{22} \end{aligned}$$

The Choi's matrix $C = (T(e_{i,j}))_{i,j}$ associated to T is

$$\begin{bmatrix} \lambda(1 + e^{-2t} \mu_n^4) & 0 & 0 & e^{-t} \\ 0 & \lambda(1 - e^{-2t}) & 0 & 0 \\ 0 & 0 & (1 - \lambda)(1 - e^{-2t}) & 0 \\ e^{-t} & 0 & 0 & (1 - \lambda)(1 + e^{-2t} \mu_n^{-4}) \end{bmatrix}$$

Since $\mu_n^4 = \frac{1-\lambda}{\lambda}$ and

$$\lambda(1 - \lambda)(1 + e^{-2t} \mu_n^4)(1 + e^{-2t} \mu_n^{-4}) - e^{-2t} = \lambda(1 - \lambda)(1 - e^{-2t})^2 \geq 0,$$

C is positive and T is completely positive.

The state preserving property follows from Proposition 3.1. \square

3.2. Main estimates. We start with the main statement.

Theorem 3.5. *Let $1 < p \leq 2$.*

$$\|P_t^\varepsilon\|_{L^p \rightarrow L^2} \leq 1 \text{ if } e^{-2t} \leq C \alpha_\mu^{4-\frac{8}{p}} (p-1)$$

for some universal constant $C > 0$, where $\alpha_\mu = \sup_{n \geq 1} \mu_n$.

Before proceeding to the proof, we collect some lemmas. The first and the most crucial one is an asymmetric version of optimal convexity inequality ([4, 8]).

Lemma 3.6. *Let $1 < p \leq 2$, $\mu \geq 1$ and $\lambda = \frac{1}{1+\mu^4}$. For any $A, B \in \mathbb{M}_n$, $n \geq 1$ we have*

$$\left(\lambda \|A + \mu^2 B\|_p^p + (1 - \lambda) \|A - \mu^{-2} B\|_p^p \right)^{\frac{2}{p}} \geq \|A\|_p^2 + C(p-1) \|B\|_p^2$$

where $C = C(\mu) = \frac{1}{3} \mu^{-4}$ for $1 < p \leq \frac{3}{4}$ and $C = \frac{1}{3} \mu^{8-\frac{16}{p}}$ for $\frac{3}{4} < p \leq 2$.

Proof. The above inequality is nothing but the contractivity of a fixed linear map from an L^p space to a L^p -valued ℓ^2 space. By a careful examination of the adjoint map we can observe that the above inequality is equivalent to the following. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $X, Y \in \mathbb{M}_n$ we have

$$(3.5) \quad \left(\lambda \|X + Y\|_q^q + (1 - \lambda) \left\| X - \frac{\lambda}{1 - \lambda} Y \right\|_q^q \right)^{\frac{2}{q}} \leq \|X\|_q^2 + \frac{q-1}{\mu^4 C} \|Y\|_q^2.$$

Since we care less about the best constant we will use the following standard argument. Let C_q be the best constant such that (3.5) is true if we replace $\frac{q-1}{\mu^4 C}$ by C_q . Then we have

$$\begin{aligned}
& \lambda \|X + Y\|_{2q}^{2q} + (1 - \lambda) \left\| X - \frac{\lambda}{1 - \lambda} Y \right\|_{2q}^{2q} \\
&= \lambda \left\| |X|^2 + |Y|^2 + X^* Y + Y^* X \right\|_q^q \\
&\quad + (1 - \lambda) \left\| |X|^2 + \left(\frac{\lambda}{1 - \lambda} \right)^2 |Y|^2 - \frac{\lambda}{1 - \lambda} (X^* Y + Y^* X) \right\|_q^q \\
&\leq \lambda \left\| |X|^2 + |Y|^2 + X^* Y + Y^* X \right\|_q^q \\
&\quad + (1 - \lambda) \left\| |X|^2 + |Y|^2 - \frac{\lambda}{1 - \lambda} (X^* Y + Y^* X) \right\|_q^q \\
&\leq \left(\left\| |X|^2 + |Y|^2 \right\|_q^2 + C_q \|X^* Y + Y^* X\|_q^2 \right)^{\frac{q}{2}} \\
&\leq \left([\|X\|_{2q}^2 + \|Y\|_{2q}^2]^2 + 4C_q \|X\|_{2q}^2 \|Y\|_{2q}^2 \right)^{\frac{q}{2}} \\
&\leq \left(\|X\|_{2q}^2 + (2C_q + 1) \|Y\|_{2q}^2 \right)^{\frac{2q}{2}}.
\end{aligned}$$

The first inequality is by monotony of the L^p norm on positive elements as $\frac{\lambda}{1-\lambda} \leq 1$.

Thus, we can conclude that $C_{2q} \leq 2C_q + 1$. Since we have $C_2 = \mu^{-4} \leq 1$, a standard interpolation argument leads us to $C_q \leq (q-1)^{1-\theta} (2q-1)^\theta$, where $2^n \leq q < 2^{n+1}$ and $\frac{1}{q} = \frac{1-\theta}{2^n} + \frac{\theta}{2^{n+1}}$. Thus, we simply get

$$C_q \leq 3(q-1),$$

which implies that the original inequality is true for

$$C = \frac{1}{3} \mu^{-4}.$$

When $n = 1$, i.e. $2 \leq q < 4$ we can get a sharper estimate. Since $C_2 = \mu^{-4}$ and $C_4 \leq 2\mu^{-4} + 1$, for $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{4}$ we have

$$C_q \leq (\mu^{-4})^{1-\theta} (2\mu^{-4} + 1)^\theta \leq 3\mu^{4-\frac{16}{q}},$$

which implies that the original inequality is true for

$$C = \frac{1}{3} \mu^{8-\frac{16}{p}}.$$

□

Lemma 3.7. For any $a, b \in \Gamma_{\langle 1, \dots, n-1 \rangle}$, with $\lambda = \frac{1}{\mu_n^4 + 1}$, we have

$$\begin{aligned}
(3.6) \quad \left\| (a + y_n d) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 &= \left(\lambda \left\| (a + \mu_n^2 d) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^p \right. \\
&\quad \left. + (1 - \lambda) \left\| (a - \mu_n^{-2} d) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^p \right)^{\frac{2}{p}} \\
&\geq \left\| a D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^2 + C(\mu)(p-1) \left\| d D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^2,
\end{aligned}$$

where $C(\mu)$ is the constant in Lemma 3.6.

Proof. It is a direct application of Proposition 3.3 and Lemma 3.6, if one notices that for any $a, d \in \Gamma_{\langle 1, \dots, n-1 \rangle}$, $\Phi(a + y_n d) = \begin{bmatrix} a + \mu_n^2 d & 0 \\ 0 & a - \mu_n^{-2} d \end{bmatrix}$. \square

Lemma 3.8. *Let $b, c \in \Gamma_{\langle 1, \dots, n-1 \rangle}$. Then, we have*

$$\left\| \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p \geq \lambda^{\frac{1}{p}} (\mu_n^2 + \mu_n^{-2})^{\frac{1}{2}} \left\| b D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p$$

and

$$\left\| \gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p \geq (1 - \lambda)^{\frac{1}{p}} (\mu_n^2 + \mu_n^{-2})^{\frac{1}{2}} \left\| c D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p.$$

Proof. By (3.6) we get

$$\begin{aligned} \left\| \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p &\geq \frac{1}{\|\gamma_n^*\|_\infty} \left\| \gamma_n^* \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p \\ &= \frac{1}{\|\gamma_n^*\|_\infty} \left\| (\mu_n^{-2} b + y_n b) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p \\ &\geq \frac{1}{\|\gamma_n^*\|_\infty} \lambda^{\frac{1}{p}} (\mu_n^2 + \mu_n^{-2}) \left\| b D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p \\ &= \lambda^{\frac{1}{p}} (\mu_n^2 + \mu_n^{-2})^{\frac{1}{2}} \left\| b D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p. \end{aligned}$$

Note that the equality in the last line holds by the fact

$$\|\gamma_n\|_\infty = \|\gamma_n^*\|_\infty = \sqrt{\mu_n^2 + \mu_n^{-2}},$$

which is a direct application of Proposition 3.3.

The estimate for $\left\| \gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p$ is similar. \square

Proof of Theorem 3.5. We follow the idea of Carlen/Lieb and Biane to use the induction on n , where $I = \{\pm 1, \dots, \pm n\}$. We assume that we have the conclusion for $n-1$ and consider the case n . Every element in $\Gamma_{\langle 1, \dots, n \rangle}$ can be uniquely expressed as

$$X = a + \gamma_n b + \gamma_n^* c + y_n d,$$

where $a, b, c, d \in \Gamma_{\langle 1, \dots, n-1 \rangle}$. Note that $\{D_{\langle n \rangle}^{\frac{1}{2}}, \gamma_n D_{\langle n \rangle}^{\frac{1}{2}}, \gamma_n^* D_{\langle n \rangle}^{\frac{1}{2}}, y_n D_{\langle n \rangle}^{\frac{1}{2}}\}$ is an orthogonal set in $L^2(\Gamma_{\langle n \rangle})$ with

$$\left\| D_{\langle n \rangle}^{\frac{1}{2}} \right\|_2 = \left\| y_n D_{\langle n \rangle}^{\frac{1}{2}} \right\|_2 = 1, \left\| \gamma_n D_{\langle n \rangle}^{\frac{1}{2}} \right\|_2 = \mu_n^{-1} \text{ and } \left\| \gamma_n^* D_{\langle n \rangle}^{\frac{1}{2}} \right\|_2 = \mu_n.$$

For example, $\left\| y_n D_{\langle n \rangle}^{\frac{1}{2}} \right\|_2^2 = \text{tr}_{\Gamma_{\langle n \rangle}}(D_{\langle n \rangle}^{\frac{1}{2}} y_n^2 D_{\langle n \rangle}^{\frac{1}{2}}) = \tau^\varepsilon(y_n^2) = \langle y_n 1, y_n 1 \rangle = 1$. Moreover, $y_n 1 = x_{-n} x_n$ so that $P_t^\varepsilon(y_n) = e^{-2t} y_n$. Thus, by applying (3) of Proposition 3.1, we get that the four terms in X are orthogonal and

$$\begin{aligned} (3.7) \quad &\left\| P_t^\varepsilon(X) D_{\langle 1, \dots, n \rangle}^{\frac{1}{2}} \right\|_2^2 \\ &= \left\| P_t^\varepsilon(a) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{2}} \right\|_2^2 + \mu_n^{-2} e^{-2t} \left\| P_t^\varepsilon(b) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{2}} \right\|_2^2 \\ &\quad + \mu_n^2 e^{-2t} \left\| P_t^\varepsilon(c) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{2}} \right\|_2^2 + e^{-4t} \left\| P_t^\varepsilon(d) D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{2}} \right\|_2^2. \end{aligned}$$

Now we estimate $\|X\|_p$. Since the map replacing γ_n into $-\gamma_n$ is a τ^ε -preserving *-isomorphism of $\Gamma_{\langle 1, \dots, n \rangle}$, Proposition 2.2 implies that

$$\left\| (a + \gamma_n b + \gamma_n^* c + y_n d) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p = \left\| (a - \gamma_n b - \gamma_n^* c + y_n d) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p.$$

By the optimal convexity inequality ([4] or [8]) we have

$$(3.8) \quad \left\| X D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 \geq \left\| (a + y_n d) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 + (p-1) \left\| (\gamma_n b + \gamma_n^* c) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 \\ = I + (p-1)II.$$

The estimate for I is Lemma 3.7. For II , note that $\gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}$ and $\gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}$ have disjoint support. Indeed, we have

$$(\gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{2}{p}} b^* \gamma_n^*) (\gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{2}{p}} c^* \gamma_n) = 0$$

and

$$(D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} b^* \gamma_n^* \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}) (D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} c^* \gamma_n \gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}) \\ = D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} b^* b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \gamma_n^* \gamma_n \gamma_n \gamma_n^* D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} c^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \\ = 0$$

by (3) of Proposition 3.1. Thus, by orthogonality and Lemma 3.8 we have

$$(3.9) \quad II = \left(\left\| \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^p + \left\| \gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^p \right)^{\frac{2}{p}} \\ \geq \left\| \gamma_n b D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 + \left\| \gamma_n^* c D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2 \\ \geq \lambda^{\frac{2}{p}} (\mu_n^2 + \mu_n^{-2}) \left\| b D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^2 + (1-\lambda)^{\frac{2}{p}} (\mu_n^2 + \mu_n^{-2}) \left\| c D_{\langle 1, \dots, n-1 \rangle}^{\frac{1}{p}} \right\|_p^2.$$

By combining (3.7), (3.8), (3.6) and (3.9) we get

$$\left\| P_t^\varepsilon(X) D_{\langle 1, \dots, n \rangle}^{\frac{1}{2}} \right\|_2^2 \leq \left\| X D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}} \right\|_p^2$$

provided that

$$e^{-2t} \leq \min\{(\mu_n^4 + 1)^{1-\frac{2}{p}}(p-1), \sqrt{C(\mu_n)}\sqrt{p-1}\},$$

where $C(\mu_n)$ is the constant in Lemma 3.6 for $\mu = \mu_n$. \square

4. APPROXIMATION BY CENTRAL LIMIT PROCEDURE

The aim of this section is to use a standard approximation procedure to go from the baby Fock model to the q -Araki-Woods algebras. Most of the arguments are easy adaptations of [16], so we will simply sketch them.

In section 3 we constructed generalized baby gaussians γ_i associated with the parameters μ_i for $1 \leq i \leq n$ by starting with the index set $I = \{\pm 1, \pm 2, \dots, \pm n\}$. In this section we apply the same construction using the increased index set

$$\tilde{I} = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(-i, -j) : 1 \leq i \leq n, 1 \leq j \leq m\},$$

so that we can get generalized baby gaussians $\gamma_{i,j}$ associated with the parameter μ_i for $1 \leq i \leq n, 1 \leq j \leq m$ and the von Neumann algebra $\Gamma_{n,m}$ generated by $\{\gamma_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Note that the “choice of sign” function ε in this case would be

$$\varepsilon : \tilde{I} \times \tilde{I} \rightarrow \{\pm 1\}$$

satisfying

$$\begin{aligned}\varepsilon((i_1, i_2), (j_1, j_2)) &= \varepsilon((j_1, j_2), (i_1, i_2)), \\ \varepsilon((i_1, i_2), (i_1, i_2)) &= -1, \\ \varepsilon((i_1, i_2), (j_1, j_2)) &= \varepsilon((|i_1|, |i_2|), (|j_1|, |j_2|)), \quad \forall (i_1, i_2), (j_1, j_2) \in \tilde{I}.\end{aligned}$$

Now we replace $\varepsilon((i_1, i_2), (j_1, j_2))$, $(i_1, i_2) \prec (j_1, j_2) \in \tilde{I}$ with a family of i.i.d. random variables with

$$P(\varepsilon((i_1, i_2), (j_1, j_2)) = -1) = \frac{1-q}{2}, \quad P(\varepsilon((i_1, i_2), (j_1, j_2)) = 1) = \frac{1+q}{2},$$

where $(i_1, i_2) \prec (j_1, j_2)$ means $i_1 < j_1$ or $i_1 = j_1, i_2 < j_2$. We set

$$s_{i,m} = \frac{1}{\sqrt{m}} \sum_{j=1}^m \gamma_{i,j}.$$

Then, the Speicher's central limit procedure ([20, 16]) tells us the following.

Proposition 4.1. *For any $*$ -polynomial Q in n non-commuting variables we have*

$$\lim_{m \rightarrow \infty} \tau^\varepsilon(Q(s_{1,m}, \dots, s_{n,m})) = \tau_q(Q(g_{q,1}, \dots, g_{q,n}))$$

for almost every ε .

Since the set of all non-commuting $*$ -polynomials is countable, we can find a choice of sign ε such that the above is true for any Q . In the sequel we fix such an ε .

Now we would like to transfer this convergence in distribution into L^p -norm convergence using Nou's ultraproduct approach ([16, Theorem 4.3, Section 5.2]). If we set $g_{i,m} = \operatorname{Re}(s_{i,m})$, $g_{-i,m} = \operatorname{Im}(s_{i,m})$ and $G_i = \operatorname{Re}(g_{q,i})$, $G_{-i} = \operatorname{Im}(g_{q,i})$, $1 \leq i \leq n$, then by Proposition 4.1 for any polynomial P in $2n$ non-commuting variables we have

$$(4.1) \quad \lim_{m \rightarrow \infty} \tau^\varepsilon(P(g_{-n,m}, \dots, g_{n,m})) = \tau_q(P(G_{-n}, \dots, G_n)).$$

We need to truncate $g_{j,m}$ to get a uniform control on the operator norms. Let $C > 0$ be a constant satisfying $\|G_j\|_{\Gamma_q} < C$ for any $|j| \leq n$. We consider the function h on \mathbb{R} with $h(x) = 1_{(-C,C)}(x)x$, $x \in \mathbb{R}$ and set $\tilde{g}_{i,m} = h(g_{i,m})$, $1 \leq i \leq n$. From [16, Lemma 5.7] and the discussion after it, we have

Proposition 4.2. *Let \mathcal{U} be a fixed free ultrafilter on \mathbb{N} , $(\mathcal{A}, \tau) = \Pi_{m, \mathcal{U}}(\Gamma_{n,m}, \tau^\varepsilon)$, and $p \in \mathcal{A}$ be the support of τ . Then we have the following normal state-preserving $*$ -isomorphism.*

$$\Theta : (\Gamma_q, \tau_q) \rightarrow (\mathcal{A}, \tau), \quad P(G_{-n}, \dots, G_n) \mapsto p \cdot (P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m}))_{m, \mathcal{U}} \cdot p,$$

where P is any polynomial in $2n$ non-commuting variables.

Then by Proposition 4.2, Proposition 2.2 and [18, Theorem 3.6] for any polynomial P in $2n$ non-commuting variables we have

$$\lim_{m, \mathcal{U}} \left\| P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m}) D_m^{\frac{1}{p}} \right\|_p = \left\| P(G_{-n}, \dots, G_n) D_q^{\frac{1}{p}} \right\|_p,$$

where D_m is the density of τ^ε restricted to $\Gamma_{n,m}$. Now we need to replace $\tilde{g}_{i,m}$ back with $g_{i,m}$.

Lemma 4.3. *Let \mathcal{U} be a fixed free ultrafilter on \mathbb{N} and $1 \leq p \leq 2$. For any polynomial P in $2n$ non-commuting variables we have*

$$\lim_{m, \mathcal{U}} \left\| P(g_{-n,m}, \dots, g_{n,m}) D_m^{\frac{1}{p}} \right\|_p = \left\| P(G_{-n}, \dots, G_n) D_q^{\frac{1}{p}} \right\|_p.$$

Proof. In the proof of [16, Lemma 5.7] it is shown that

$$\lim_{m \rightarrow \infty} \left| \tau^\varepsilon(\tilde{g}_{n,j_1} \cdots \tilde{g}_{n,j_{k-1}}(g_{n,j_k} - \tilde{g}_{n,j_k})g_{n,j_{k+1}} \cdots g_{n,j_l}) \right| = 0$$

for any indices j_1, \dots, j_l and $1 \leq k \leq l$. By taking involution inside the functional τ^ε we also get

$$\lim_{m \rightarrow \infty} \left| \tau^\varepsilon(g_{n,j_1} \cdots g_{n,j_{k-1}}(g_{n,j_k} - \tilde{g}_{n,j_k})\tilde{g}_{n,j_{k+1}} \cdots \tilde{g}_{n,j_l}) \right| = 0.$$

If we apply the above limits repeatedly, then we have

$$(4.2) \quad \lim_{m \rightarrow \infty} \left| \tau^\varepsilon(\tilde{g}_{n,j_1} \cdots \tilde{g}_{n,j_{k-1}}g_{n,j_k} \cdots g_{n,j_l}) - \tau^\varepsilon(g_{n,j_1} \cdots g_{n,j_{k-1}}g_{n,j_k} \cdots g_{n,j_l}) \right| = 0$$

and

$$(4.3) \quad \lim_{m \rightarrow \infty} \left| \tau^\varepsilon(g_{n,j_1} \cdots g_{n,j_{k-1}}\tilde{g}_{n,j_k} \cdots \tilde{g}_{n,j_l}) - \tau^\varepsilon(\tilde{g}_{n,j_1} \cdots \tilde{g}_{n,j_{k-1}}\tilde{g}_{n,j_k} \cdots \tilde{g}_{n,j_l}) \right| = 0.$$

Now we consider any polynomial P in $2n$ non-commuting variables, then we have

$$\begin{aligned} & \left\| P(g_{-n,m}, \dots, g_{n,m})D_m^{\frac{1}{2}} - P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m})D_m^{\frac{1}{2}} \right\|_2^2 \\ &= \left| \tau^\varepsilon(P^*P - \tilde{P}^*P - P^*\tilde{P} + \tilde{P}^*\tilde{P}) \right| \leq \left| \tau^\varepsilon(P^*P - \tilde{P}^*P) \right| + \left| \tau^\varepsilon(P^*\tilde{P} - \tilde{P}^*\tilde{P}) \right|, \end{aligned}$$

where P and \tilde{P} denote $P(g_{-n,m}, \dots, g_{n,m})$ and $P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m})$, respectively. Since $P^*P - \tilde{P}^*P$ and $P^*\tilde{P} - \tilde{P}^*\tilde{P}$ are linear combinations of the terms of the forms

$$\tilde{g}_{n,j_1} \cdots \tilde{g}_{n,j_{k-1}}g_{n,j_k} \cdots g_{n,j_l} - g_{n,j_1} \cdots g_{n,j_{k-1}}g_{n,j_k} \cdots g_{n,j_l}$$

and

$$g_{n,j_1} \cdots g_{n,j_{k-1}}\tilde{g}_{n,j_k} \cdots \tilde{g}_{n,j_l} - \tilde{g}_{n,j_1} \cdots \tilde{g}_{n,j_{k-1}}\tilde{g}_{n,j_k} \cdots \tilde{g}_{n,j_l},$$

respectively, (4.2) and (4.3) imply that

$$\lim_{m \rightarrow \infty} \left\| P(g_{-n,m}, \dots, g_{n,m})D_m^{\frac{1}{2}} - P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m})D_m^{\frac{1}{2}} \right\|_2 = 0.$$

Since $L^2(\tau^\varepsilon)$ embeds into $L^p(\tau^\varepsilon)$ contractively we get

$$\lim_{m \rightarrow \infty} \left\| P(g_{-n,m}, \dots, g_{n,m})D_m^{\frac{1}{p}} - P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m})D_m^{\frac{1}{p}} \right\|_p = 0,$$

so that

$$\begin{aligned} \lim_{m, \mathcal{U}} \left\| P(g_{-n,m}, \dots, g_{n,m})D_m^{\frac{1}{p}} \right\|_p &= \lim_{m, \mathcal{U}} \left\| P(\tilde{g}_{-n,m}, \dots, \tilde{g}_{n,m})D_m^{\frac{1}{p}} \right\|_p \\ &= \left\| P(G_{-n}, \dots, G_n)D_q^{\frac{1}{p}} \right\|_p. \end{aligned}$$

□

Remark 4.4. We can extend Lemma 4.3 for the case $2 < p < \infty$.

The following lemma is a non-tracial version of [3, Lemma 5].

Lemma 4.5. *For any $*$ -polynomial Q in n non-commuting variables and $1 \leq p \leq 2$ we have*

$$\lim_{m \rightarrow \infty} \left\| P_t^\varepsilon(Q(s_{1,m}, \dots, s_{n,m}))D_m^{\frac{1}{p}} \right\|_p = \left\| P_t^q(Q(g_{q,1}, \dots, g_{q,n}))D_q^{\frac{1}{p}} \right\|_p.$$

Proof. The proof is essentially the same as [3, Lemma 5], so that we omit it. Note that we need Lemma 4.3 for the conclusion. □

Proof of Theorem 2.3. By a standard density argument it is enough to consider the case $\dim \mathcal{H} = n$. Then for any $*$ -polynomial Q in n non-commuting variables and a fixed free ultrafilter \mathcal{U} on \mathbb{N} we have

$$\left\| P_t^q(Q(g_{q,1}, \dots, g_{q,n})) D_q^{\frac{1}{2}} \right\|_2 = \lim_{m, \mathcal{U}} \left\| P_t^\varepsilon(Q(s_{1,m}, \dots, s_{n,m})) D_m^{\frac{1}{2}} \right\|_2$$

by Lemma 4.5. Theorem 3.5 implies that

$$\left\| P_t^\varepsilon(Q(s_{1,m}, \dots, s_{n,m})) D_m^{\frac{1}{2}} \right\|_2 \leq \left\| Q(s_{1,m}, \dots, s_{n,m}) D_m^{\frac{1}{p}} \right\|_p$$

if $e^{-2t} \leq C \alpha_\mu^{4-\frac{8}{p}} (p-1)$, where C is the constant in Theorem 3.5. Applying Lemma 4.3 we get

$$\left\| P_t^q(Q(g_{q,1}, \dots, g_{q,n})) D_q^{\frac{1}{2}} \right\|_2 \leq \left\| P_t^q(Q(g_{q,1}, \dots, g_{q,n})) D_q^{\frac{1}{p}} \right\|_p.$$

□

Remark 4.6. For the most general case of $\Gamma_q(H_\mathbb{R}, (U_t))$ we use the discretization argument in [16, section 6], where the following embedding has been established.

For a fixed free ultrafilter \mathcal{U} on \mathbb{N} we have the following normal state-preserving $*$ -isomorphism.

$$\Theta : (\Gamma_q(H_\mathbb{R}, (U_t)), \tau_q) \rightarrow \Pi_{n, \mathcal{U}}(\Gamma_n, \tau_n), \quad G(e_i) \mapsto p \cdot (G_n(e_i))_{n, \mathcal{U}} \cdot p,$$

where (Γ_n, τ_n) 's are almost periodic q -Araki-Woods algebras, $G(e_i)$, $G_n(e_i)$'s are corresponding gaussians and $p \in \Pi_{n, \mathcal{U}} \Gamma_n$ be the support of $\Pi_{n, \mathcal{U}} \tau_n$.

Then, the same ultraproduct argument as above proves Theorem 1.3.

5. 1-DIMENSIONAL ESTIMATE

We consider the “only if” direction by examining 1-dimensional behavior as usual. We start by an estimate of the L^p -norm of g_i , the q -gaussian with the parameter μ_i . As $g_i^* g_i$ is in the centralizer of φ .

$$\|g_i D_{\langle i \rangle}^{\frac{1}{p}}\|_p = \|D_{\langle i \rangle}^{\frac{1}{p}} g_i^* g_i D_{\langle i \rangle}^{\frac{1}{p}}\|_{p/2}^{\frac{1}{2}} = \varphi((g_i^* g_i)^{p/2})^{\frac{1}{p}}.$$

The self-adjoint element $y = g_i^* g_i$ can be seen as a commutative random variable in some probability space with measure induced by φ . It is well known that q -creations are bounded for $-1 \leq q < 1$ ([6, Lemma 4]) so $\|y\|_\infty \sim \mu_i^2$ with constants depending only on q . Moreover, we have already seen that $\|y\|_1 = \frac{1}{\mu_i^2}$ and $\|y\|_2 \sim 1$.

It follows from the Hölder inequality, that $\|y\|_p = \varphi((g_i^* g_i)^p)^{\frac{1}{p}} \sim \mu_i^{2-4/p}$ with constants depending only on q . So we conclude that for $p \geq 2$

$$\|g_i D_{\langle i \rangle}^{\frac{1}{p}}\|_p \sim \mu_i^{1-\frac{4}{p}}.$$

By duality if P_t^q can be extended to a contraction from $L^p(\Gamma_q)$ into $L^2(\Gamma_q)$, then it can also be extended from $L^2(\Gamma_q)$ into $L^{p'}(\Gamma_q)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, so

$$e^{-t} \leq \mu_i^{2-\frac{4}{p}}.$$

That is

Theorem 5.1. *Suppose that $\alpha_\mu = \sup_n \mu_n = \infty$, then P_t^q can not be extended to a contraction from $L^p(\Gamma_q)$ into $L^2(\Gamma_q)$ for any $1 \leq p < 2$.*

We give a more precise estimate for $p \rightarrow 1$. Let $n \in \mathbb{N}$, and we set $a(\varepsilon) = (1 + \varepsilon g_i) D_{\langle i \rangle}^{\frac{1}{2n}}$, $\varepsilon > 0$. Then we have

$$|a(\varepsilon)|^{2n} = (D_{\langle i \rangle}^{\frac{1}{2n}} (1 + \varepsilon g_i + \varepsilon g_i^* + \varepsilon^2 g_i^* g_i) D_{\langle i \rangle}^{\frac{1}{2n}})^n.$$

If we expand the right hand side, then we get

$$\begin{aligned}
& D_{\langle i \rangle} + \varepsilon(D_{\langle i \rangle}^{\frac{1}{2n}} g_i D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-1}{n}} + \cdots + D_{\langle i \rangle}^{\frac{n-1}{n}} D_{\langle i \rangle}^{\frac{1}{2n}} g_i D_{\langle i \rangle}^{\frac{1}{2n}}) \\
& + \varepsilon(D_{\langle i \rangle}^{\frac{1}{2n}} g_i^* D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-1}{n}} + \cdots + D_{\langle i \rangle}^{\frac{n-1}{n}} D_{\langle i \rangle}^{\frac{1}{2n}} g_i^* D_{\langle i \rangle}^{\frac{1}{2n}}) \\
& + n\varepsilon^2 D_{\langle i \rangle} g_i^* g_i \\
& + \varepsilon^2(D_{\langle i \rangle}^{\frac{1}{2n}} g_i^* D_{\langle i \rangle}^{\frac{1}{n}} g_i D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-2}{n}} + \cdots \\
& + D_{\langle i \rangle}^{\frac{1}{2n}} g_i D_{\langle i \rangle}^{\frac{1}{n}} g_i^* D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-2}{n}} + \cdots \\
& + D_{\langle i \rangle}^{\frac{1}{2n}} g_i^* D_{\langle i \rangle}^{\frac{2}{n}} g_i D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-3}{n}} + \cdots \\
& + D_{\langle i \rangle}^{\frac{1}{2n}} g_i D_{\langle i \rangle}^{\frac{2}{n}} g_i^* D_{\langle i \rangle}^{\frac{1}{2n}} D_{\langle i \rangle}^{\frac{n-3}{n}} + \cdots) \\
& + o(\varepsilon^2) D_{\langle i \rangle}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\text{tr}(|a(\varepsilon)|^{2n}) &= 1 + n\varepsilon^2 \mu_i^{-2} + \varepsilon^2 \left(\sum_{k=1}^{n-1} (n-k) \mu_i^{\frac{4}{n}k-2} + \sum_{k=1}^{n-1} k \mu_i^{-\frac{4}{n}(n-k)+2} \right) + o(\varepsilon^2) \\
&= 1 + n\varepsilon^2 \sum_{k=0}^{n-1} \mu_i^{\frac{4}{n}k-2} + o(\varepsilon^2) = 1 + n\varepsilon^2 \mu_i^{-2} \frac{\mu_i^{\frac{4}{n}} - 1}{\mu_i^{\frac{4}{n}} - 1} + o(\varepsilon^2),
\end{aligned}$$

so that

$$\begin{aligned}
\|P_q^t(a(\varepsilon))\|_{2n} &= \|a(e^{-t}\varepsilon)\|_{2n} \\
&= \left(1 + ne^{-2t} \varepsilon^2 \mu_i^{-2} \frac{\mu_i^{\frac{4}{n}} - 1}{\mu_i^{\frac{4}{n}} - 1} + o(\varepsilon^2) \right)^{\frac{1}{2n}} \\
&= 1 + \frac{e^{-2t} \varepsilon^2}{2} \mu_i^{-2} \frac{\mu_i^{\frac{4}{n}} - 1}{\mu_i^{\frac{4}{n}} - 1} + o(\varepsilon^2) \\
&\geq 1 + \frac{n}{2} e^{-2t} \varepsilon^2 \mu_i^{2-\frac{4}{n}} + o(\varepsilon^2).
\end{aligned}$$

Consequently, $\|P_q^t(a(\varepsilon))\|_{2n} \leq \|a(\varepsilon)\|_2$ implies that

$$1 + \frac{n}{2} e^{-2t} \varepsilon^2 \mu_i^{2-\frac{4}{n}} + o(\varepsilon^2) \leq 1 + \frac{\varepsilon^2}{2} \mu_i^{-2} + o(\varepsilon^2),$$

which means

$$(5.1) \quad e^{-2t} \leq \frac{1}{n} \mu_i^{-4+\frac{4}{n}} \leq 2\mu_i^{-4+\frac{8}{2n}} \frac{1}{2n-1}$$

by taking $\varepsilon \rightarrow 0$. By duality we get the following.

Theorem 5.2. *Let $\frac{1}{p} = 1 - \frac{1}{2n}$, $n(\geq 2) \in \mathbb{N}$. Then $\|P_q^t\|_{L^p \rightarrow L^2} \leq 1$ implies that*

$$e^{-2t} \leq 2\alpha_\mu^{4-\frac{8}{p}}(p-1).$$

If we turn back to the baby Fock model, this one dimensional estimate can be extended for all $1 < p < 2$. That is, Theorem 3.5 is optimal.

Proposition 5.3. *Let $d \in \mathbb{M}_n$ be an invertible self-adjoint matrix, and $g \in \mathbb{M}_n$, such that $dg = \lambda gd$ for some $\lambda > 1$. Then, for any $p > 2$,*

$$\|(1 + \varepsilon g)d\|_p^p = \text{Tr } d^p + \varepsilon^2 \left(\left(\frac{p}{2} + c_{p,\lambda} \right) \text{Tr } d^p g^* g + c_{p,\frac{1}{\lambda}} \text{Tr } d^p g g^* \right) + O(\varepsilon^2),$$

where

$$c_{p,\lambda} = \frac{\lambda^p - 1}{(\lambda^2 - 1)(1 - \frac{1}{\lambda^2})} - \frac{p\lambda^2}{2(\lambda^2 - 1)}.$$

Proof. We use the well known fact that on positive definite matrices, the map $f : x \mapsto x^{p/2}$ is C^∞ . Moreover, its derivative at x can be expressed easily in terms of the spectral decomposition of x and divided differences of f ; if $x = \sum sp_s$ is the spectral decomposition of x , then for $h \in \mathbb{M}_n^{sa}$:

$$\text{diff}_x f \cdot h = \sum_{s,t} f_1(s,t) p_s h p_t$$

$$\text{diff}_x^2 f \cdot (h, h) = 2 \cdot \sum_{s,t,u} f_2(s,t,u) p_s h p_t h p_u$$

where

$$f_1(a,b) = \begin{cases} \frac{f(a)-f(b)}{a-b} & \text{if } a \neq b \\ f'(a) & \text{if } a = b \end{cases}$$

$$f_2(a,b,c) = \begin{cases} \frac{f_1(a,c)-f_1(b,c)}{a-b} & \text{if } a \neq b \\ \lim_{h \rightarrow 0} \frac{f_1(a+h,c)-f_1(a,c)}{h} & \text{if } a = b \end{cases}$$

Under the trace, for our choice of f :

$$\text{Tr}(\text{diff}_x f \cdot h) = \frac{p}{2} \text{Tr} x^{p/2-1} h$$

$$\text{Tr}(\text{diff}_x^2 f \cdot (h, h)) = 2 \cdot \text{Tr} \left(\sum_{s,t} f_2(s,t,s) p_s h p_t h \right).$$

We want the expansion at the second order in ε of

$$\|(1 + \varepsilon g)d\|_p^p = \text{Tr}(d^2 + \varepsilon d(g + g^*)d + \varepsilon^2 dg^*gd)^{p/2}.$$

By the above formula, with $x = d^2$, the first order term is $\frac{p}{2} \text{Tr} d^p (g + g^*) = 0$ because of the commutation relation as $\lambda \neq 1$.

By the Taylor expansion, the second order term has two contributions, one from the first derivative, the other coming from one half the second one. The first is given by $\frac{p}{2} \text{Tr} d^p g^*g$. The second is more involved; let $d = \sum_{\alpha \in \sigma(d)} \alpha p_\alpha$ be its spectral decomposition, we get

$$\begin{aligned} A &= \text{Tr} \left(\sum_{\alpha, \beta \in \sigma(d)} f_2(\alpha^2, \beta^2, \alpha^2) p_\alpha d(g + g^*) d p_\beta d(g + g^*) d \right) \\ &= \text{Tr} \left(\sum_{\alpha, \beta \in \sigma(d)} (\alpha\beta)^2 f_2(\alpha^2, \beta^2, \alpha^2) p_\alpha (g + g^*) p_\beta (g + g^*) p_\alpha \right). \end{aligned}$$

The relation $dg = \lambda gd$ gives that $P(d)g = gP(\lambda d)$ for any polynomial P . It yields $p_\alpha g = gp_{\frac{\alpha}{\lambda}}$, where $p_{\frac{\alpha}{\lambda}}$ is zero if $\frac{\alpha}{\lambda}$ is not a eigenvalue of d . In particular,

$$p_\alpha (g + g^*) p_\beta (g + g^*) p_\alpha = g^2 p_{\frac{\alpha}{\lambda^2}} p_{\frac{\alpha}{\lambda}} p_\alpha + gg^* p_\alpha p_\beta p_\alpha + g^* g p_\alpha p_{\frac{\alpha}{\lambda}} p_\alpha + g^{*2} p_{\alpha\lambda^2} p_\alpha p_\beta p_\alpha.$$

Thus

$$A = \text{Tr} \left(\sum_{\alpha \in \sigma(d)} \frac{\alpha^4}{\lambda^2} f_2(\alpha^2, \frac{\alpha^2}{\lambda^2}, \alpha^2) p_\alpha g g^* \right) + \text{Tr} \left(\sum_{\alpha \in \sigma(d)} \alpha^4 \lambda^2 f_2(\alpha^2, \alpha^2 \lambda^2, \alpha^2) p_\alpha g^* g \right).$$

Then,

$$\frac{\alpha^4}{\lambda^2} f_2(\alpha^2, \frac{\alpha^2}{\lambda^2}, \alpha^2) = \alpha^p \left(\frac{p}{2(\lambda^2 - 1)} - \frac{1 - \frac{1}{\lambda^p}}{(\lambda^2 - 1)(1 - \frac{1}{\lambda^2})} \right) = \alpha^p c_{p, \frac{1}{\lambda}}.$$

Finally,

$$A = c_{p, \frac{1}{\lambda}} \text{Tr} d^p g g^* + c_{p, \lambda} \text{Tr} d^p g^* g.$$

□

To conclude, using the notation of Section 3, we apply it with $d = D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}$, $g = \gamma_n$, $\lambda = \mu_n^{\frac{4}{p}}$. Recall that $\text{Tr } d^p g^* g = \mu_n^{-2}$ and $\text{Tr } d^p g g^* = \mu_n^2$,

$$\begin{aligned} \|(1 + \varepsilon \gamma_n) D_{\langle 1, \dots, n \rangle}^{\frac{1}{p}}\|_p^p &= 1 + \varepsilon^2 \left(\frac{p}{2\mu_n^2} - \frac{p\mu_n^{\frac{s}{p}}}{2\mu_n^2(\mu_n^{\frac{s}{p}} - 1)} + \frac{p\mu_n^2}{2(\mu_n^{\frac{s}{p}} - 1)} \right) + O(\varepsilon^2) \\ &= 1 + \varepsilon^2 \cdot \frac{p}{2\mu_n^2} \cdot \frac{\mu_n^{\frac{4}{p}} - 1}{\mu_n^{\frac{s}{p}} - 1} + O(\varepsilon^2). \end{aligned}$$

Then, the conclusion about the optimality follows as above.

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